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Asymmetric Maximal and Minimal Open Sets

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Abstract. We introduce the notions of maximal and minimal open sets in bitopological spaces and obtain some properties of them. In contrary to maximal and minimal open sets in topological spaces, we observe that maximal and minimal open sets in bitopological spaces behave differently. The maximal and minimal open sets in a bitopological space under the operations of union and intersection respectively sometimes become slightly different types of maximal and minimal open sets in that bitopological space. We also obtain results concerning an asymmetric minimal open set on a subspace of a bitopological space.

1. Introduction

Generalizations of existing topological ideas for genuine reasons are imperative fields of study to topologists. Noticing the asymmetric nature of quasi-metric spaces, Kelly [3] introduced the notion of bitopological spaces: a nonempty set X endowed with two distinct topologies $\mathscr{P}_1$ and $\mathscr{P}_2$ on X is called a bitopological space and it is denoted by $(X, \mathscr{P}_1, \mathscr{P}_2)$. The existence of a bitopological space is very much natural and hence generalizations of some prevailing topological concepts in bitopological spaces are demands of time. Nakaoka and Oda [7] introduced and studied the concept of minimal open sets in a topological space. Dualizing the concept of minimal open sets, Nakaoka and Oda [6] introduced and studied the notion of maximal open sets in a topological space. Generalizing the notion of maximal open sets, we introduce and study maximal open sets (Definition 2.1) in bitopological spaces. Then dualizing the concepts of maximal open sets in a bitopological space, we introduce and study minimal open sets (Definition 3.1) in a bitopological space. Maximal and minimal open sets in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ are asymmetric in the sense that maximality and minimality of a nontrivial $(\mathscr{P}_i)$open set $(i \in \{1, 2\})$ is defined with respect to a $(\mathscr{P}_j)$open set $(j \in \{1, 2\}, j \neq i)$. In contrary to maximal and minimal open sets in a topological space, maximal and minimal open sets in a bitopological space

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are variant e.g., Theorem 2.3 and Theorem 3.4 under the operations of union and intersection respectively.

Benchalli et al. [1] introduced and studied the notions of pairwise maximal and minimal open sets which are similar to the notions of $(\mathscr{P}_i, \mathscr{P}_j)$maximal open sets (Definition 2.1) and $(\mathscr{P}_i, \mathscr{P}_j)$minimal open sets (Definition 3.1) for some $i \in \{1, 2\}$ $(j \in \{1, 2\}, j \neq i)$ respectively introduced in this paper. We introduce and study asymmetric nature of pairwise maximal and minimal open sets of a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ along with $(\mathscr{P}_i, \mathscr{P}_j)$maximal and $(\mathscr{P}_i, \mathscr{P}_j)$minimal open sets which are totally different from the properties of pairwise maximal and minimal open sets due to Benchalli et al. [1]. Ghour and Azaizeh [2] introduced and studied s-minimal open sets in bitopological spaces. A set $A \subset X$ is called an s-minimal open set [2] in the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ if A is a minimal open set in $(X, \mathscr{P}_i)$ for each $i \in \{1, 2\}$. We see that pairwise minimal open sets in the sense of this paper are equivalent to s-minimal open sets in bitopological spaces.

For a subset A of a topological space $(X, \mathscr{T})$, $(\mathscr{T})Cl(A)$ and $(\mathscr{T})Int(A)$ denote the closure and interior of A respectively with respect to the topological space $(X, \mathscr{T})$. For $A \subset X$, we write 'A is $(\mathscr{T})$open' to mean '$A \in \mathscr{T}$' where $\mathscr{T}$ is a topology on X. For a topological space $(X, \mathscr{T})$ and $A \subset X$, we write $(A, \mathscr{T}_A)$ to denote the subspace on A of $(X, \mathscr{T})$. So the relative bitopological space for $(X, \mathscr{P}_1, \mathscr{P}_2)$ corresponding to $A \subset X$ is $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$. Unless otherwise mentioned, X stands for the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$. Always $i, j \in \{1, 2\}$ and whenever i, j appear together, $j \neq i$. Throughout the paper, N denotes the set of natural numbers and R, the set of real numbers.

2. Asymmetric Maximal Open Sets

Definition 2.1. A nontrivial $(\mathscr{P}_i)$open set A of a bitopological space X is said to be $(\mathscr{P}_i, \mathscr{P}_j)$maximal open if B is a $(\mathscr{P}_j)$open set of X containing A, then either $B = A$ or $B = X$. A is said to be absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open if B is a $(\mathscr{P}_j)$open set of X containing A, then $B = X$.

A subset A of a bitopological space X is said to be pairwise maximal open if A is both $(\mathscr{P}_1, \mathscr{P}_2)$maximal open and $(\mathscr{P}_2, \mathscr{P}_1)$maximal open.

A pairwise maximal open set of a bitopological space is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. An absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open set can not be a pairwise maximal open set.

A set may not be pairwise maximal open even if the set is both $(\mathscr{P}_j)$open and $(\mathscr{P}_i, \mathscr{P}_j)$maximal open $(i, j \in \{1, 2\}, j \neq i)$.

Example 2.1. For any $a \in R$, we define

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\}, \\ \mathscr{P}_2 &= \{\emptyset, R, (-\infty, a), [a, \infty)\}. \end{aligned}$$

In the bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$, $(-\infty, a)$ is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open and it is also $(\mathscr{P}_2)$open. But it is not $(\mathscr{P}_2, \mathscr{P}_1)$maximal open and hence $(-\infty, a)$ is not pairwise maximal open.

Example 2.2 (Mukharjee and Bose [5])**.** Let b be a fixed real number. We define

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, R\} \cup \{(-\infty, b], (b, \infty)\}, \\ \mathscr{P}_2 &= \{\emptyset, R\} \cup (b, \infty)\} \bigcup \left\{ \left(b + \frac{1}{n}, \infty \right) \mid n \in N \right\}. \end{aligned}$$

In the bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$, (b, ∞) is pairwise maximal open and $\left(b + \frac{1}{n}, \infty\right)$ is $(\mathscr{P}_2, \mathscr{P}_1)$maximal open for no $n \in N$.

Example 2.3 (Mukharjee [4])**.** For any $a \in R$, we define

$$\begin{aligned} \mathscr{P}_1 &= \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\}, \\ \mathscr{P}_2 &= \{\emptyset, R, \{a\}, (a, \infty), [a, \infty)\}. \end{aligned}$$

In the bitopological space $(R, \mathscr{P}_1, \mathscr{P}_2)$, $(-\infty, a)$ is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open but it is not maximal open in $(R, \mathscr{P}_1)$.

So it follows from Example 2.3 that for $i \neq j$, a $(\mathscr{P}_i, \mathscr{P}_j)$maximal open set in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ may not be maximal open in $(X, \mathscr{P}_i)$ for some $i \in \{1, 2\}$.

Theorem 2.1. *If A is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1, 2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open, then either $A \cup B = X$ or $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open or A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ with $B \subset A$.*

Proof. We note that A is nontrivial $(\mathscr{P}_i)$open for some $i \in \{1, 2\}$. Two cases may arise: $A \cup B = X$ or $A \cup B \neq \emptyset, X$. We need to consider only the case $A \cup B \neq X$. Suppose there exists a $(\mathscr{P}_j)$open set $U \neq X, A \cup B$ such that $A \cup B \subset U$. Then we get $A \subset A \cup B \subset U$. Since A is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open and $U \neq X$, we have $A = U \Rightarrow A \cup B = U$, a contradiction to our assumption $U \neq A \cup B$. So we have $U = X$ or $U = A \cup B$ which imply that $A \cup B$ is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open. If $U = A \cup B \neq X$, then $A \cup B$ is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and so by $(\mathscr{P}_i, \mathscr{P}_j)$maximal openness of A, we get $A = A \cup B$ which implies A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and $B \subset A$. □

Corollary 2.1. *If A is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1, 2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open, then either $A \cup B = X$ or $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open.*

Proof. Since A is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1, 2\}, j \neq i$ in X, A is $(\mathscr{P}_i)$open and A is not $(\mathscr{P}_j)$open. So the results follow by Theorem 2.1. □

Theorem 2.2. *If $A \notin \mathscr{P}_j$ is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1,2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open, then either $A \cup B = X$ or $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open.*

Proof. Similar to the proof of Theorem 2.1. □

Theorem 2.3. *If A, B are distinct $(\mathscr{P}_i, \mathscr{P}_j)$maximal open sets in X for some $i \in \{1,2\}, j \neq i$, then either $A \cup B = X$ or $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open.*

Proof. Proceeding like the proof of Theorem 2.1, we see that if there exists a $(\mathscr{P}_j)$open set U such that $A \cup B \subset U$, then $U = X$ or $U = A \cup B$. If $U = A \cup B \neq X$, then $A \cup B$ is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ and so by $(\mathscr{P}_i, \mathscr{P}_j)$maximal openness of A, we get $A = A \cup B$ which implies A is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ and $B \subset A$. Since B is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open and $A \neq X$, we get $A = B$ which is not possible by hypothesis. Similarly, considering $B \subset A \cup B \subset U$, we may have $U = X$ or $A = B$. Hence $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open if $A \cup B \neq X$. □

Corollary 2.2. *If A, B are distinct absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open sets in X for some $i \in \{1,2\}, j \neq i$, then either $A \cup B = X$ or $A \cup B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open.*

Theorem 2.4. *Let A be a $(\mathscr{P}_i, \mathscr{P}_j)$maximal open set for some $i \in \{1,2\}$ and $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ in X. Then either A is the only such set in X or the union of such sets in X is X.*

Proof. Let A, B be $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1,2\}$ and $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ in X. Then $A \cup B$ is also $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ in X. As $A \subset A \cup B$ and A is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open, we have $A = A \cup B \Rightarrow B \subset A$ or $A \cup B = X$. Similarly, for B, we get $A \subset B$ or $A \cup B = X$. $B \subset A$ and $A \cup B = X$ imply that $A = X$ which is not possible. Similarly, $A \subset B$ together with $A \cup B = X$ is not possible. The only possible cases are $B \subset A, A \subset B \Rightarrow A = B$ and $A \cup B = X$. □

Theorem 2.5. *If A is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, B is $(\mathscr{P}_2, \mathscr{P}_1)$maximal open and $A \cup B$ is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$, then either $A \cup B = X$ or $A = B$.*

Proof. Note that A is nontrivial $(\mathscr{P}_1)$open and B is nontrivial $(\mathscr{P}_2)$open. Since $A \subset A \cup B$ and A is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, considering $A \cup B$ as $(\mathscr{P}_2)$open, we get either $A = A \cup B$ which implies $B \subset A$ or $A \cup B = X$. Considering $A \cup B$ as $(\mathscr{P}_1)$open, by $B \subset A \cup B$ and $(\mathscr{P}_2, \mathscr{P}_1)$maximal openness of B, we have either $B = A \cup B$ which implies $A \subset B$ or $A \cup B = X$. The only feasible possibilities are $A = B$ and $A \cup B = X$. □

Corollary 2.3. *If A is absolutely $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, B is absolutely $(\mathscr{P}_2, \mathscr{P}_1)$maximal open and $A \cup B$ is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$, then $A \cup B = X$.*

Proof. Since A is absolutely $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, A is $(\mathscr{P}_1)$open but A is not $(\mathscr{P}_2)$open. If $A = B$, then A should be $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, since B is $(\mathscr{P}_2)$open. □

Corollary 2.4. *The pairwise maximal open sets in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is either unique or their union is X.*

Proof. Let A, B be two pairwise maximal open sets of X. A, B are nontrivial $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and hence $A \cup B$ is also $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. So the results follow by Theorem 2.5. □

Definition 2.2 (Pervin [8])**.** A bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ is said to be connected if X cannot be expressed as the union of two nonempty disjoint sets A and B such that one of A, B is $(\mathscr{P}_1)$open and other is $(\mathscr{P}_2)$open. X is disconnected if X is so expressible.

Corollary 2.5. *If A is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, B is $(\mathscr{P}_2, \mathscr{P}_1)$maximal open with $A \cap B = \emptyset$ and $A \cup B$ is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, then X is disconnected.*

Proof. It follows that A is $(\mathscr{P}_1)$open and B is $(\mathscr{P}_2)$open. As $A \cap B = \emptyset$, we have $A \neq B$. So by Theorem 2.5, we get $A \cup B = X$. □

Corollary 2.6. *If A is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open, B is $(\mathscr{P}_2, \mathscr{P}_1)$maximal open with $A \neq B$, $A \cap B$ is $(\mathscr{P}_i)$closed for some $i \in \{1, 2\}$ and $A \cup B$ is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, then X is disconnected.*

Proof. We see that A is $(\mathscr{P}_1)$open and B is $(\mathscr{P}_2)$open. By Corollary 2.5, X is disconnected if $A \cap B = \emptyset$. We suppose that $A \cap B \neq \emptyset$. As $A \neq B$, we have by Theorem 2.5, $A \cup B = X$. If $A \cap B$ is $(\mathscr{P}_1)$closed, we write $G = A - A \cap B$, $H = B$. If $A \cap B$ is $(\mathscr{P}_2)$closed, we write $G = A$, $H = B - A \cap B$. In either case G is $(\mathscr{P}_1)$open, H is $(\mathscr{P}_2)$open with $G \cap H = \emptyset$ and $G \cup H = X$. □

Lemma 2.1. *If a subset A of X is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and $(\mathscr{P}_i, \mathscr{P}_j)$maximal open for some $i \in \{1, 2\}$, then A is maximal open in $(X, \mathscr{P}_j)$.*

Proof. Let U be a $(\mathscr{P}_j)$open set such that $A \subset U$. As A is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open, we have $A = U$ or $U = X$. Since A is $(\mathscr{P}_j)$open, it follows from the definition that A is maximal open in $(X, \mathscr{P}_j)$. □

Theorem 2.6. *A subset of $(X, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise maximal open iff it is maximal open in $(X, \mathscr{P}_i)$ for each $i \in \{1, 2\}$.*

Proof. Firstly, suppose that A is maximal open in $(X, \mathscr{P}_i)$ for each $i \in \{1, 2\}$. So A is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. As A is maximal open in $(X, \mathscr{P}_i)$ for each $i \in \{1, 2\}$, we have $A = U$ or $U = X$ for any $(\mathscr{P}_i)$open set U containing A. Considering A as $(\mathscr{P}_1)$open, we do not get a $(\mathscr{P}_2)$open set U such that $A \subset U$ and $U \neq A, X$. So A is $(\mathscr{P}_1, \mathscr{P}_2)$maximal open. Similarly, A is $(\mathscr{P}_2, \mathscr{P}_1)$maximal open. Thus A is pairwise maximal open.

Conversely, a pairwise maximal open set of $(X, \mathscr{P}_1, \mathscr{P}_2)$ is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. By Lemma 2.1, the result follows. □

Theorem 2.7. *If A is both $(\mathscr{P}_j)$open and $(\mathscr{P}_i, \mathscr{P}_j)$maximal open $(i, j \in \{1, 2\}, i \neq j)$ in X, then either $(\mathscr{P}_j)Cl(A) = X$ or $(\mathscr{P}_j)Cl(A) = A$.*

Proof. We show that either no $x \in X - A$ is a $(\mathscr{P}_j)$limit point of A or each $x \in X - A$ is a $(\mathscr{P}_j)$limit point of A. Let G be a $(\mathscr{P}_j)$open nbd of $x \in X - A$. Then we have $A \subset A \cup G$. Since $A \cup G$ is $(\mathscr{P}_j)$open and A is $(\mathscr{P}_i, \mathscr{P}_j)$maximal open, we may have Case I: $A = A \cup G$ and Case II: $A \cup G = X$.

Case I: For $x \in X - A$, we consider a $(\mathscr{P}_j)$open nbd G of x such that $A = A \cup G$. Now $A = A \cup G \Rightarrow x \in G \subset A$, a contradiction to $x \in X - A$. In other words, no point of $X - A$ is a $(\mathscr{P}_j)$limit point of A. So $(\mathscr{P}_j)Cl(A) = A$.

Case II: For each $(\mathscr{P}_j)$open nbd G of $x \in X - A$, we consider $A \cup G = X$. Now $A \cup G = X \Rightarrow X - A \subset G$. If $G = A$, then $A \cup G = X \Rightarrow A = X$ which is impossible. Also if $A \cap G = \emptyset$, then $A = X - G$ which means that A is $(\mathscr{P}_j)$closed i.e. $(\mathscr{P}_j)Cl(A) = A$. So we now suppose that no $(\mathscr{P}_j)$open nbd G of x is identical to $A, X - A$. Then $X - A \subset G$ and $G \neq A, X - A$ implies that $G \cap A \neq \emptyset$. So each $x \in X - A$ is a $(\mathscr{P}_j)$limit point of A and hence $(\mathscr{P}_j)Cl(A) = X$. □

Corollary 2.7. *If A is pairwise maximal open, then either $(\mathscr{P}_i)Cl(A) = X$ or $(\mathscr{P}_i)Cl(A) = A$ for each $i \in \{1, 2\}$.*

Proof. Since A is pairwise maximal open, A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$. Hence the corollary follows by Theorem 2.7. □

Corollary 2.8. *If A is both $(\mathscr{P}_j)$open and $(\mathscr{P}_i, \mathscr{P}_j)$maximal open $(i, j \in \{1, 2\}, i \neq j)$ in X, then either $(\mathscr{P}_j)Int(X - A) = \emptyset$ or $(\mathscr{P}_j)Int(X - A) = X - A$.*

Proof. Follows by Theorem 2.7. □

3. ASYMMETRIC MINIMAL OPEN SETS

We introduce the concept of pairwise minimal open sets by dualizing the concept of pairwise maximal open sets.

Definition 3.1. A nontrivial $(\mathscr{P}_i)$open set A of a bitopological space X is said to be $(\mathscr{P}_i, \mathscr{P}_j)$minimal open if B is a $(\mathscr{P}_j)$open set of X contained in A, then either $B = A$ or $B = \emptyset$. A is said to be absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open if B is a $(\mathscr{P}_j)$open set of X contained in A, then $B = \emptyset$.

A subset A of a bitopological space X is said to be pairwise minimal open if A is both $(\mathscr{P}_1, \mathscr{P}_2)$minimal open and $(\mathscr{P}_2, \mathscr{P}_1)$minimal open. An absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open set can not be a pairwise minimal open set.

A pairwise minimal open set of a bitopological space is nontrivial $(\mathscr{P}_i)$open for each $i \in \{1,2\}$.

A set may not be pairwise minimal open even if the set is both $(\mathscr{P}_j)$open and $(\mathscr{P}_i, \mathscr{P}_j)$minimal open $(i, j \in \{1,2\}, j \neq i)$. For, we consider Example 2.2. In the bitopological space of Example 2.2, (b, ∞) is $(\mathscr{P}_2, \mathscr{P}_1)$minimal open and nontrivial $(\mathscr{P}_i)$open for each $i \in \{1,2\}$. But it is not pairwise minimal open.

In the bitopological space of Example 2.1, $(-\infty, a)$ is pairwise minimal open. In the bitopological space of Example 2.2, (b, ∞) is $(\mathscr{P}_2, \mathscr{P}_1)$minimal open but it is not minimal open in $(R, \mathscr{P}_2)$. So it follows that for $i \neq j$, a $(\mathscr{P}_i, \mathscr{P}_j)$minimal open set in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ may not be minimal open in $(X, \mathscr{P}_i)$ for some $i \in \{1,2\}$.

By dualizing some earlier results, we have the results from Theorem 3.1 to Theorem 3.7. The proofs of these results are omitted as the proofs are similar to the proofs of corresponding results already established.

Theorem 3.1. *If A is $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1,2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open with $A \cap B \neq \emptyset$, then either $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open or A is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ with $A \subset B$.*

Corollary 3.1. *If A is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1,2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open with $A \cap B \neq \emptyset$, then $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open.*

Theorem 3.2. *If $A \notin \mathscr{P}_j$ is $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1,2\}, j \neq i$ in X and B is $(\mathscr{P}_i)$open with $A \cap B \neq \emptyset$, then $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open.*

Theorem 3.3. *If A, B are distinct $(\mathscr{P}_i, \mathscr{P}_j)$minimal open sets in X for some $i \in \{1,2\}$, $j \neq i$ with $A \cap B \neq \emptyset$, then $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open.*

Theorem 3.4. *If A, B are $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1,2\}, j \neq i$ with $A \cap B \neq \emptyset$, then either $A = B$ or $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open.*

Corollary 3.2. *If A, B are absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1,2\}, j \neq i$ with $A \cap B \neq \emptyset$, then either $A = B$ or $A \cap B$ is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open.*

Theorem 3.5. *If A, B are $(\mathscr{P}_i, \mathscr{P}_j)$minimal open sets for some $i \in \{1,2\}$ and $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ in X with $A \cap B \neq \emptyset$, then $A = B$.*

Theorem 3.6. *If A is $(\mathscr{P}_1, \mathscr{P}_2)$minimal open, B is $(\mathscr{P}_2, \mathscr{P}_1)$minimal open and $A \cap B \neq \emptyset$ is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$, then $A = B$.*

Corollary 3.3. *There can not exists a pair of distinct absolutely $(\mathscr{P}_1, \mathscr{P}_2)$minimal open set A and absolutely $(\mathscr{P}_2, \mathscr{P}_1)$minimal open set B with $A \cap B \neq \emptyset$ is $(\mathscr{P}_i)$open for each $i \in \{1,2\}$ in a bitopological space.*

Corollary 3.4. *If A, B are pairwise minimal open sets in a bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ with $A \cap B \neq \emptyset$, then $A = B$.*

Lemma 3.1. *If a subset A of X is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and $(\mathscr{P}_i, \mathscr{P}_j)$minimal open for some $i \in \{1, 2\}$, then A is minimal open in $(X, \mathscr{P}_j)$.*

Theorem 3.7. *A subset of $(X, \mathscr{P}_1, \mathscr{P}_2)$ is pairwise minimal open iff it is minimal open in $(X, \mathscr{P}_i)$ for each $i \in \{1, 2\}$.*

In view of Theorem 3.7, it follows that a pairwise minimal open set is equivalent to an s-minimal open set due to Ghour and Azaizeh [2].

Suppose that $(X, \mathscr{P}_1, \mathscr{P}_2)$ is a bitopological space having only one nontrivial $(\mathscr{P}_1)$open set A_1 and only one nontrivial $(\mathscr{P}_2)$open set A_2 with $A_1 \cap A_2 = \emptyset$. Then for each $i \in \{1, 2\}$, A_i is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$maximal open as well as absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open, $j \in \{1, 2\}, j \neq i$. In this case, the bitopological space $(X, \mathscr{P}_1, \mathscr{P}_2)$ become unstable in the sense that two topologies become identical if A_i, $i \in \{1, 2\}$ is either pairwise maximal open or pairwise minimal open. Again suppose that $(X, \mathscr{P}_1, \mathscr{P}_2)$ is a bitopological space having only one nontrivial $(\mathscr{P}_1)$open set A and only one nontrivial $(\mathscr{P}_2)$open set B with $B \subsetneqq A$. Then A is absolutely $(\mathscr{P}_1, \mathscr{P}_2)$maximal open and B is absolutely $(\mathscr{P}_2, \mathscr{P}_1)$minimal open, $i, j \in \{1, 2\}, j \neq i$. Note that in this case also, the bitopological space become unstable if $A = B$.

Theorem 3.8. *Let A be $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and G be $(\mathscr{P}_i)$open for some $i \in \{1, 2\}$ in $(X, \mathscr{P}_1, \mathscr{P}_2)$ such that $A \cap G \neq \emptyset, G$. Then $A \cap G$ is absolutely $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$minimal open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$ if G is $(\mathscr{P}_i, \mathscr{P}_j)$minimal open in $(X, \mathscr{P}_1, \mathscr{P}_2)$.*

Proof. If possible, suppose that there exists a $(\mathscr{P}_{jA})$open set $U \neq \emptyset$ in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$ such that $U \subsetneqq A \cap G$. So we have $U \subsetneqq A \cap G \subset G$. Since A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, U is $(\mathscr{P}_j)$open in X. By $(\mathscr{P}_i, \mathscr{P}_j)$minimal openness of G, we get $U = G$ which implies that $A \cap G = G$, a contradiction to our assumption. □

Theorem 3.9. *Let A be $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$ and G be $(\mathscr{P}_i)$open for some $i \in \{1, 2\}$ in $(X, \mathscr{P}_1, \mathscr{P}_2)$. Then $A \cap G$ is absolutely $(\mathscr{P}_{iA}, \mathscr{P}_{jA})$minimal open in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$ if G is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open in $(X, \mathscr{P}_1, \mathscr{P}_2)$.*

Proof. If possible, suppose that there exists a $(\mathscr{P}_{jA})$open set U in $(A, \mathscr{P}_{1A}, \mathscr{P}_{2A})$ such that $U \subset A \cap G$. So we have $U \subset A \cap G \subset G$. Since A is $(\mathscr{P}_i)$open for each $i \in \{1, 2\}$, U is $(\mathscr{P}_j)$open in X. As G is absolutely $(\mathscr{P}_i, \mathscr{P}_j)$minimal open, we get $U = \emptyset$. □

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